Improved Inclusion-Exclusion Identities and Bonferroni Inequalities with Applications to Reliability Analysis of Coherent Systems

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Chapter 1

Introduction and Overview

Many problems in combinatorics, number theory, probability theory, reliability theory and statistics can be solved by applying a unifying method, which is known as the *principle of inclusion-exclusion*. The principle of inclusion-exclusion expresses the indicator function of a union of finitely many sets as an alternating sum of indicator functions of their intersections. More precisely, for any finite family of sets $\{A_v\}_{v\in V}$ the classical principle of inclusion-exclusion states that

$$\chi\left(\bigcup_{v\in V} A_v\right) \;=\; \sum_{\substack{I\subseteq V\\ I\neq\emptyset}} (-1)^{|I|-1}\,\chi\left(\bigcap_{i\in I} A_i\right)\,, \tag{1.1}$$

where $\chi(A)$ denotes the indicator function of A with respect to some ground set Ω, that is, $\chi(A)(\omega)=1$ if $\omega\in A$, and $\chi(A)(\omega)=0$ if $\omega\in\Omega\setminus A$. Equivalently, (1.1) can be expressed as $\chi\big(\bigcap_{v\in V}\complement A_v\big) = \sum_{I\subseteq V}(-1)^{|I|}\chi\big(\bigcap_{i\in I} A_i\big)$, where $\complement A_v$ denotes the complement of A_v in Ω and $\bigcap_{i\in\emptyset} A_i = \Omega$. A proof by induction on the number of sets is a common task in undergraduate courses. Usually, the A_v's are measurable with respect to some finite measure μ on a σ-field of subsets of Ω. Integration of the indicator function identity (1.1) with respect to μ then gives

$$\mu\left(\bigcup_{v\in V} A_v\right) \;=\; \sum_{\substack{I\subseteq V\\ I\neq\emptyset}} (-1)^{|I|-1}\,\mu\left(\bigcap_{i\in I} A_i\right)\,, \tag{1.2}$$

which now expresses the measure of a union of finitely many sets as an alternating sum of measures of their intersections. The step leading from (1.1) to (1.2) is referred to as the *method of indicators* [GS96b]. Naturally, two special cases are of particular interest, namely the case where μ is the counting measure on the power set of Ω and the case where μ is a probability measure on a σ-field of subsets of Ω. These special cases are sometimes attributed to Sylvester (1883) and Poincaré (1896), although the second edition of Montmort's book "Essai d'Analyse sur les Jeux de Hazard", which appeared in 1714, already contains an

implicit use of the method, based on a letter by N. Bernoulli in 1710. A first explicit description of the method was given by Da Silva (1854). For references to these sources and additional historical notes, we refer to Takács [Tak67].

Since the classical identities (1.1) and (1.2) contain $2^{|V|}-1$ terms and intersections of up to $|V|$ sets, one often resorts to inequalities like that of Boole [Boo54]:

$$\chi\left(\bigcup_{v\in V} A_v\right) \;\le\; \sum_{i\in V} \chi(A_i)\,. \tag{1.3}$$

A more general result, first discovered by Ch. Jordan [Jor27] and later by Bonferroni [Bon36], states that for any finite family of sets $\{A_v\}_{v\in V}$ and any $r \in \mathbb{N}$,

$$\chi\left(\bigcup_{v\in V} A_v\right) \;\le\; \sum_{\substack{I\subseteq V\\ 0<|I|\le r}} (-1)^{|I|-1}\,\chi\left(\bigcap_{i\in I} A_i\right) \qquad (r \text{ odd}), \tag{1.4}$$

$$\chi\left(\bigcup_{v\in V} A_v\right) \;\ge\; \sum_{\substack{I\subseteq V\\ 0<|I|\le r}} (-1)^{|I|-1}\,\chi\left(\bigcap_{i\in I} A_i\right) \qquad (r \text{ even}). \tag{1.5}$$

Nowadays, these inequalities are usually referred to as *Bonferroni inequalities*. Again, there is no real restriction in using indicator functions rather than measures, since these inequalities can be integrated with respect to any finite measure μ (e.g., a probability measure) on any σ-field containing the sets A_v, $v \in V$.

Numerous extensions of the classical Bonferroni inequalities (1.4) and (1.5) were established in the second half of the 20th century. An excellent survey of the various results, applications and methods of proof is given in the recent book of Galambos and Simonelli [GS96b]. The following inequalities due to Galambos [Gal75], which are valid for any finite collection of sets $\{A_v\}_{v\in V}$, improve (1.4) and (1.5) by including additional terms based on the $(r+1)$-subsets of V:

$$\chi\left(\bigcup_{v\in V} A_v\right) \;\le\; \sum_{\substack{I\subseteq V\\ 0<|I|\le r}} (-1)^{|I|-1}\,\chi\left(\bigcap_{i\in I} A_i\right) - \frac{r+1}{|V|}\sum_{\substack{I\subseteq V\\ |I|=r+1}} \chi\left(\bigcap_{i\in I} A_i\right) \qquad (r \text{ odd}),$$

$$\chi\left(\bigcup_{v\in V} A_v\right) \;\ge\; \sum_{\substack{I\subseteq V\\ 0<|I|\le r}} (-1)^{|I|-1}\,\chi\left(\bigcap_{i\in I} A_i\right) + \frac{r+1}{|V|}\sum_{\substack{I\subseteq V\\ |I|=r+1}} \chi\left(\bigcap_{i\in I} A_i\right) \qquad (r \text{ even}).$$

A related result due to Tomescu [Tom86] states that

$$\chi\left(\bigcup_{v\in V} A_v\right) \;\le\; \sum_{\substack{I\subseteq V\\ 0<|I|\le r}} (-1)^{|I|-1}\,\chi\left(\bigcap_{i\in I} A_i\right) - \sum_{I\in\mathcal{E}_{r+1}} \chi\left(\bigcap_{i\in I} A_i\right) \qquad (r \text{ odd}),$$

$$\chi\left(\bigcup_{v\in V} A_v\right) \;\ge\; \sum_{\substack{I\subseteq V\\ 0<|I|\le r}} (-1)^{|I|-1}\,\chi\left(\bigcap_{i\in I} A_i\right) + \sum_{I\in\mathcal{E}_{r+1}} \chi\left(\bigcap_{i\in I} A_i\right) \qquad (r \text{ even}),$$

where $\mathcal{E}_{r+1}$ is the edge-set of a so-called $(r+1)$-hypertree.

Inequalities for the measure or indicator function of a union which are valid for any finite collection of sets like the preceding ones are frequently referred to as *Bonferroni-type inequalities* [GS96a, GS96b] or as inequalities of *Bonferroni-Galambos type* [MS85, Măr89, TX89]. A new inequality of Bonferroni-Galambos type based on chordal graphs will be established in Section 4.3 of this thesis.

The main part of this work deals with improved inclusion-exclusion identities and improved Bonferroni inequalities that require the collection of sets to satisfy some structural restrictions. Examples of such well-structured collections of sets arise in some problems of statistical inference [NW92, NW97], combinatorial reliability theory [Doh98b, Doh99c] and chromatic graph theory [Doh99a, Doh99d]. We shall mainly be interested in inclusion-exclusion identities of the form

$$\chi\left(\bigcup_{v\in V} A_v\right) \;=\; \sum_{I\in\mathcal{S}} (-1)^{|I|-1}\, \chi\left(\bigcap_{i\in I} A_i\right)$$

and inequalities of type

$$\chi\left(\bigcup_{v\in V} A_v\right) \;\leq\; \sum_{\substack{I\in\mathcal{S}\\ |I|\leq r}} (-1)^{|I|-1}\, \chi\left(\bigcap_{i\in I} A_i\right) \qquad (r \text{ odd}), \tag{1.6}$$

$$\chi\left(\bigcup_{v\in V} A_v\right) \;\geq\; \sum_{\substack{I\in\mathcal{S}\\ |I|\leq r}} (-1)^{|I|-1}\, \chi\left(\bigcap_{i\in I} A_i\right) \qquad (r \text{ even}), \tag{1.7}$$

where $\mathcal{S}$ is a restricted set of non-empty subsets of V, and where (1.6) and (1.7) are at least as sharp as their classical counterparts (1.4) and (1.5). A first straightforward approach is to account only for non-empty subsets I of V satisfying $\bigcap_{i\in I} A_i \neq \emptyset$. In fact, Lozinskii [Loz92] demonstrates that a skillful implementation of this approach leads to a reduction of the average running time of the standard inclusion-exclusion algorithm for counting satisfying assignments of propositional formulae in conjunctive normal form. In the present thesis, however, we are interested in more subtle improvements that arise from logical dependencies of the sets involved. Consider for instance the five sets A_1–A_5, whose Venn diagram is shown in Figure 1.1. The classical inclusion-exclusion identity for the indicator function of the union of these sets gives a sum of $2^5 - 1 = 31$ terms, many of which are equal with opposite sign. After cancelling out, we are left with

$$\begin{aligned}\chi(A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5) \;=\; & \chi(A_1) + \chi(A_2) + \chi(A_3) + \chi(A_4) + \chi(A_5) \\ & - \chi(A_1 \cap A_2) - \chi(A_2 \cap A_3) - \chi(A_3 \cap A_4) - \chi(A_4 \cap A_5)\,,\end{aligned}$$

which contains only 9 terms. Our purpose is to predict such cancellations and thus to obtain shorter inclusion-exclusion identities and sharper Bonferroni inequalities

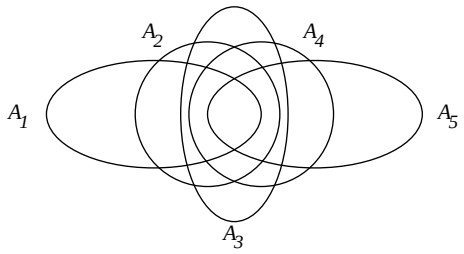

Figure 1.1: A Venn diagram of five sets.

for the indicator function of a union. Although the improved inclusion-exclusion identities follow from the associated improved Bonferroni inequalities, we prefer to treat the identities separately, since they can be proved in an elementary combinatorial way, whereas the inequalities require a topological proof.

The thesis is organized as follows: In Chapter 2 we introduce some terminology on graphs and partially ordered sets that will be repeatedly used in later chapters. In Chapter 3 we bring in some relevant structures and establish several improvements of the classical inclusion-exclusion identity. Several results from the literature such as the semilattice sieve of Narushima [Nar74, Nar82] and the tree sieve of Naiman and Wynn [NW92] are rediscovered in a unified way. In Chapter 4 we give a detailed survey of the recent theory of abstract tubes, which was initiated by Naiman and Wynn [NW92, NW97], and establish some improved Bonferroni inequalities based on the results of this theory. The chapter concludes with a new Bonferroni-Galambos-type inequality based on chordal graphs, itself subsuming several other inequalities. In Chapter 5 the results are applied to reliability analysis of coherent systems such as communication networks, k-out-of-n systems and consecutive k-out-of-n systems. Among other things we rediscover Shier's recursive algorithm and semilattice expression for the reliability of a coherent system [Shi88, Shi91] and establish some related Bonferroni inequalities. We then turn our attention to reliability covering problems and identify a comprehensive class of hypergraphs for which the coverage probability can be computed in polynomial time. In Chapter 6, which is devoted to miscellaneous topics, we give a new and simplified proof of Whitney's broken circuit theorem on the chromatic polynomial of a graph [Whi32] and establish some new inequalities on that polynomial. The results are then extended to a new two-variable polynomial that generalizes both the chromatic polynomial and the independence polynomial of a graph. We finally draw similar conclusions for the Tutte polynomial, the characteristic polynomial and the β invariant of a matroid, the Euler characteristic of an abstract simplicial complex and the Möbius function of a partially ordered set.

Chapter 2

Preliminaries

In this chapter we review some common notions on graphs and posets. Some particular notions are defined later when they are first needed. Readers with background in graph theory and lattice theory may want to skip this chapter.

2.1 Graphs

Unless stated otherwise, all graphs in this thesis are finite, undirected and without loops or multiple edges. When directed edges or loops and multiple edges are allowed, we respectively speak of a digraph or multigraph rather than a graph.

Definition 2.1.1 A *graph* is a pair $G = (V, E)$ where $V = V(G)$ is a finite set, whose elements are the *vertices* of the graph, and where $E = E(G)$ is a set of two-element subsets of V, whose elements are the *edges* of the graph. We write $n(G)$ and $m(G)$ to denote the number of vertices and edges of G, respectively. If $e = \{x, y\}$ is an edge of G, then e is said to *join* x and y, and x and y are called *adjacent in* G. G is *complete* if any two distinct vertices of G are adjacent in G. The *neighborhood* $N_G(v)$ of a vertex v of G is the set of all vertices w of G such that v and w are adjacent in G; the cardinality of $N_G(v)$ is called the *degree* of v. A *subgraph* of G is a graph G' such that $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$. With any subset I of $V(G)$ we associate the *vertex-induced subgraph* $G[I] := (I, \{e \in E(G) \mid e \subseteq I\})$. Similarly, with any subset J of $E(G)$ we associate the edge-subgraph $G[J] := \left(\bigcup J, J\right)$ where $\bigcup J$ is a shorthand for $\bigcup_{j \in J} j$. A subset X of $V(G)$ is *independent* or *stable* if $G[X]$ is edgeless. A *clique* of G is a subset X of $V(G)$ such that $G[X]$ is complete. The *clique number* of G is the maximum cardinality of a clique of G. A vertex $v \in V(G)$ is a *simplicial vertex* of G if its neighborhood is a clique. An *path between x and y* or *x, y-path* in G is a sequence $(v_1, \ldots, v_k)$ where $k \geq 1$ and where $v_1, \ldots, v_k$ are distinct vertices of G such that $x = v_1$, $y = v_k$ and $\{v_i, v_{i+1}\} \in E(G)$ for $i = 1, \ldots, k-1$. Two paths $(v_1, \ldots, v_k)$ and $(w_1, \ldots, w_l)$ are *independent* if $v_i \neq w_j$ for $i = 2, \ldots, k-1$ and $j = 2, \ldots, l-1$. A *cycle* of G is a sequence $(v_1, \ldots, v_k, v_1)$ where $k \geq 3$

and where $v_1, \ldots, v_k$ are distinct vertices of G such that $\{v_i, v_{i+1}\} \in E(G)$ for $i = 1, \ldots, k-1$ and $\{v_k, v_1\} \in E(G)$. The *length* of a path $(v_1, \ldots, v_k)$ resp. cycle $(v_1, \ldots, v_k, v_1)$ is k. The *girth* of a graph which is not cycle-free is the length of a shortest cycle in the graph. Throughout, paths and cycles of G are viewed as subgraphs of G. A *chord* of a path P of G is an edge of G joining two vertices of P that are not adjacent in P; similarly for cycles. G is *chordal* or *triangulated* if any cycle of G of length greater than three has a chord. G is *connected* if there is a path between any pair of vertices of G. A *connected component* of G is a maximal connected subgraph of G. The number of connected components of G is denoted by $c(G)$. G is *biconnected* if there are at least two independent paths between any pair of vertices of G. A maximal biconnected subgraph of G is a *block* of G. G is a *block graph* if every block of G is a complete graph.

A *tree* is a connected graph without cycles. Any connected subgraph of a tree is a *subtree* of that tree. Vertices of degree 0 or 1 in a tree are called *leaves*. A subgraph G' of a graph G is a *spanning tree* of G if G' is a tree and $V(G) = V(G')$.

Two graphs $G = (V, E)$ and $G' = (V', E')$ are *isomorphic* if there is a bijective mapping $\phi : V \to V'$ such that $\{v, w\} \in E(G)$ if and only if $\{\phi(v), \phi(w)\} \in E(G')$.

The *join* $G_1 * G_2$ of two vertex-disjoint graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is defined as $G_1 * G_2 := (V_1 \cup V_2, E_1 \cup E_2 \cup \{\{v_1, v_2\} \mid v_1 \in V_1, v_2 \in V_2\})$. If G_1 and G_2 are not vertex-disjoint, then in the preceding definition they are replaced by vertex-disjoint isomorphic copies G_1' and G_2'. Thus, the join $G_1 * G_2$ is unique up to isomorphism. Finally, for each $n \in \mathbb{N}$ we use K_n, L_n and P_n to denote the complete graph, the edgeless graph and the path on n vertices, respectively.

2.2 Posets

Our terminology on posets is standard and agrees with that of Grätzer [Gra98].

Definition 2.2.1 Let P be a set. A *partial ordering relation* on P is a binary relation which is reflexive, antisymmetric and transitive. Given such a relation, we refer to P as a *partially ordered set* or *poset*. If there are no ambiguities we use $\leq$ to denote the partial ordering relation on P and write $a < b$ if $a \leq b$ and $a \neq b$ for any $a, b \in P$. Two elements $a, b \in P$ are called *comparable* if $a \leq b$ or $b \leq a$; otherwise, they are called *incomparable*. A subset C of P is a *chain* resp. *antichain* of P if any two elements of C are comparable resp. incomparable. In the particular case where P itself is a chain the partial ordering relation on P is called *linear*. The *length* of a chain C of P is one less than the cardinality of C. For any $p \in P$ the length of a longest chain extending upward to p is called the *height* of p. The *height* of P is the maximum height of an element of P. P is called *lower-finite* resp. *upper-finite* if $\{x \in P \mid x \leq p\}$ resp. $\{x \in P \mid x \geq p\}$ is finite for any $p \in P$. A *lower bound* of a subset X of P is an element $p \in P$ such that $p \leq x$ for any $x \in X$. Dually, an *upper bound* of a subset X of P

is an element $p \in P$ such that $p \geq x$ for any $x \in X$. A *lower semilattice* is a partially ordered set P such that P contains a greatest lower bound of any non-empty finite subset X of P, which is called the *infimum* or *meet* of X and abbreviated to $\inf X$ or $\bigwedge X$. Dually, an *upper semilattice* is a partially ordered set P such that P contains a least upper bound of any non-empty finite subset X of P, which is called the *supremum* or *join* of X and abbreviated to $\sup X$ or $\bigvee X$. The *infimum* and *supremum* of any two elements $x, y \in P$ are denoted by $x \wedge y$ and $x \vee y$, respectively. An element p of a lower semilattice P is called *meet-irreducible* if $p = x \wedge y$ implies $p = x$ or $p = y$. Dually, an element p of an upper semilattice P is called *join-irreducible* if $p = x \vee y$ implies $p = x$ or $p = y$. A *lower* resp. *upper subsemilattice* of a lower resp. upper semilattice P is a subset X of P which is closed under $\wedge$ resp. $\vee$. The lower resp. upper subsemilattice of P *generated by a subset* X *of* P is the intersection of all lower resp. upper subsemilattices of P which include X as a subset. A *lattice* is a partially ordered set P which is both a lower and an upper semilattice. We write $L = [\hat{0}, \hat{1}]$ to signify that L has a least element $\hat{0}$ and greatest element $\hat{1}$. A bijective mapping ϕ from a lattice L to a lattice L' is a *lattice isomorphism* if $\phi(x \wedge y) = \phi(x) \wedge \phi(y)$ and $\phi(x \vee y) = \phi(x) \vee \phi(y)$ for any $x, y \in L$.

Usually, a partially ordered set P is represented as a *Hasse diagram* in which elements of P are represented by points in the plane and points associated with distinct elements x and y of P are joined by a line segment ascending from x to y if $x < y$ and there is no $p \in P$ lying strictly between x and y. Thus, relations implied by transitivity are not explicitly shown. Figure 6.3 on page 104 shows a Hasse diagram for a set of five elements a, b, c, d, e where $d < b, e$ and $b < a, c$.

Chapter 3

Improved Inclusion-Exclusion Identities

In this chapter we establish improvements of the classical inclusion-exclusion identity based on closure operators (having the unique basis property) and kernel operators. In a unified way we thus rediscover several results from the literature such as the tree sieve of Naiman and Wynn [NW92] and the semilattice sieve of Narushima [Nar74]. We then establish two recursive schemes for the probability of a union, which have applications in the context of system reliability analysis. Throughout, we use $\mathcal{P}(V)$ to denote the power set of any set V, that is, the set of all subsets of V, and $\mathcal{P}^*(V)$ to denote the set of all non-empty subsets of V.

3.1 Improvements based on closure operators

In what follows it is intuitive to imagine that c is the convex hull operator in $\mathbb{R}^d$.

Definition 3.1.1 Let V be a set. A *closure operator* on V is a mapping c from the power set of V into itself such that for any subsets X and Y of V,

$$\begin{aligned} &\text{(i)} \quad X \subseteq c(X) \quad \text{(extensionality)},\\ &\text{(ii)} \quad X \subseteq Y \Rightarrow c(X) \subseteq c(Y) \quad \text{(monotonicity)},\\ &\text{(iii)} \quad c(c(X)) = c(X) \quad \text{(idempotence)}. \end{aligned}$$

If c is a closure operator on V, then a subset X of V is referred to as *c-closed* if $c(X) = X$ and as *c-free* if all subsets of X are c-closed. A *c-basis* of X is a minimal subset B of X such that $c(B) = X$. If there are no ambiguities, we also write *closed* instead of c-closed, *free* instead of c-free, and *basis* instead of c-basis.

The following proposition characterizes the free sets by means of their bases.

Proposition 3.1.2 [Doh00b] *Let V be a finite set, and let c be a closure operator on V. Then, any subset J of V is free if and only if it is a basis of itself.*

Proof. Trivially, if J is free, then J is a basis of itself. Subsequently, the opposite direction is proved by contraposition. Assume that J is not free, that is, $K \subset J$ for some non-closed set K. If J is not closed, then it is not a basis of itself, and we are done. Thus, we may assume that J is closed. For each $k \in c(K) \setminus K$ we find that $k \in c(K) = c(K \setminus \{k\}) \subseteq c(J \setminus \{k\}) \subseteq c(J) = J$ and hence, $J \subseteq c(J \setminus \{k\} \cup \{k\}) \subseteq c(c(J \setminus \{k\}) \cup \{k\}) = c(c(J \setminus \{k\})) = c(J \setminus \{k\}) \subseteq c(J) = J$. Therefore, $k \in J$ and $c(J \setminus \{k\}) = J$, whence J is not a basis of itself. □

The following definition is due to Edelman and Jamison [EJ85].

Definition 3.1.3 A *convex geometry* is a pair (V, c) consisting of a finite set V and a closure operator c on V such that $c(\emptyset) = \emptyset$ and such that any c-closed subset of V has a unique c-basis.

The most prominent example of a convex geometry is the following:

Example 3.1.4 [EJ85] Let V be a finite set of points in $\mathbb{R}^d$, and for any subset $X = \{x_1, \ldots, x_n\}$ of V let $\mathrm{conv}(X)$ denote the convex hull of X, that is,

$$\mathrm{conv}(X) \;:=\; \left\{ \sum_{i=1}^{n} t_i x_i \;\middle|\; t_1, \ldots, t_n \geq 0 \text{ and } \sum_{i=1}^{n} t_i = 1 \right\} . \tag{3.1}$$

Then, by $c(X) := \mathrm{conv}(X) \cap V$ a closure operator on V is defined. By the Minkowski-Krein-Milman theorem, any c-closed subset X of V has a unique c-basis, consisting of the vertices of $\mathrm{conv}(X)$. Thus, (V, c) is a convex geometry.

Some further examples associated with graphs and semilattices follow.

Example 3.1.5 [EJ85] For any tree $G = (V, E)$ and any subset X of V define

$$c(X) \;:=\; \bigcup_{x,y \in X} \{z \in V \mid z \text{ is on the unique path between } x \text{ and } y\} .$$

Then, a subset X of V is c-closed if and only if $G[X]$ is a subtree of G, and c-free if and only if X is an edge or a singleton, that is, if $X = \{v, w\}$ for some $\{v, w\} \in E$ or $X = \{v\}$ for some $v \in V$. Since the leaves of $G[X]$ constitute a unique c-basis for any c-closed subset X of V, (V, c) is a convex geometry.

Example 3.1.6 [EJ85] Let $G = (V, E)$ be a connected block graph, and for any subset X of V let $c(X)$ be the smallest (with respect to inclusion) superset of X that induces a connected subgraph of G. Then, a subset X of V is c-closed if and only if $G[X]$ is connected, and c-free if and only if X is a clique of G, that is, if $G[X]$ is complete. Since the simplicial vertices of $G[X]$ constitute a unique c-basis for any c-closed subset X of V, we conclude that (V, c) is a convex geometry.

Example 3.1.7 [EJ85] Let V be a finite upper (resp. lower) semilattice, and for any subset X of V let $c(X)$ be the upper (resp. lower) subsemilattice of V which is generated by X. Then, the c-closed subsets of V are the upper (resp. lower) subsemilattices of V, while the c-free subsets of V are the chains of V. Since any c-closed subset X of V has a unique c-basis, namely the set of its join-irreducibles (resp. meet-irreducibles), we are again faced with a convex geometry (V, c).

Although not mentioned by Edelman and Jamison [EJ85], the following proposition generalizes a result of Narushima [Nar74, Nar77] on semilattices.

Proposition 3.1.8 [EJ85] *For any closed set J in a convex geometry (V, c),*

$$\sum_{\substack{I\subseteq J\\ c(I)=J}} (-1)^{|I|} \;=\; \begin{cases} (-1)^{|J|} & \textit{if } J \textit{ is free},\\ 0 & \textit{otherwise}. \end{cases}$$

Subsequently, we give our own proof of Proposition 3.1.8. It strongly generalizes Narushima's proof [Nar74, Nar77] for the semilattice case (Example 3.1.7).

Proof. Let J_0 be the unique basis of J. Then, $c(I) = J$ iff $J_0 \subseteq I \subseteq J$. Hence,

$$\sum_{\substack{I\subseteq J\\ c(I)=J}} (-1)^{|I|} \;=\; \begin{cases} (-1)^{|J|} & \text{if } J_0 = J,\\ 0 & \text{otherwise}. \end{cases}$$

From Proposition 3.1.2 it follows that $J_0 = J$ if and only if J is free. □

The following proposition will be used later to derive the main result of this section. It may have applications not only to inclusion-exclusion.

Proposition 3.1.9 [Doh99b] *Let (V, c) be a convex geometry, and let g be a mapping from the power set of V into an abelian group such that $g = g \circ c$. Then,*

$$\sum_{I\subseteq V} (-1)^{|I|}\, g(I) \;=\; \sum_{\substack{J\subseteq V\\ J \text{ free}}} (-1)^{|J|}\, g(J)\,.$$

Proof. By the requirements, $g(I) = g(c(I))$ for any subset I of V. Therefore,

$$\sum_{I\subseteq V} (-1)^{|I|}\, g(I) \;=\; \sum_{I\subseteq V} (-1)^{|I|}\, g(c(I)) \;=\; \sum_{\substack{J\subseteq V\\ c(J)=J}} \sum_{\substack{I\subseteq J\\ c(I)=J}} (-1)^{|I|}\, g(J)\,.$$

Now, by applying Proposition 3.1.8 the statement immediately follows. □

The following corollary contains an unpublished result due to Lawrence.

Corollary 3.1.10 [EJ85] *Let (V, c) be a convex geometry where $V \neq \emptyset$. Then,*

$$\sum_{\substack{J \subseteq V \\ J \text{ free}}} (-1)^{|J|} \;=\; 0\,.$$

Proof. For any $I \subseteq V$ define $g(I) := 1$ and apply Proposition 3.1.9. □

Although we do not need the following two corollaries, we state them since we feel that they are interesting in their own right.

Corollary 3.1.11 [Doh99b] *In any convex geometry (V, c) there are exactly*

$$\sum_{I \subseteq V} (-1)^{|c(I) \setminus I|}$$

free sets.

Proof. For any $I \subseteq V$ define $g(I) := (-1)^{|c(I)|}$ and apply Proposition 3.1.9. □

Corollary 3.1.12 [Doh99b] *Let (V, c) be a convex geometry. Then,*

$$\sum_{I \subseteq V} (-1)^{|I|}\,|c(I)| \;=\; \sum_{\substack{J \subseteq V \\ J \text{ free}}} (-1)^{|J|}\,|J|\,. \tag{3.2}$$

Proof. For any $I \subseteq V$ define $g(I) := |c(I)|$ and apply Proposition 3.1.9. □

Remark. For the convex geometry of Example 3.1.4, where c is derived from the convex hull operator in $\mathbb{R}^d$, Corollary 3.1.12 specializes to a recent result of Gordon [Gor97]. A more recent result of Edelman and Reiner [ER00] states that either side of (3.2) agrees in absolute value with the number of points in V which are in the interior of $\mathrm{conv}(V)$ if $|V| > 1$. This settles a conjecture of Ahrens, Gordon and McMahon [AGM99], who previously gave a proof for $d = 2$.

We continue with a further preliminary proposition.

Proposition 3.1.13 [Doh00b] *Let $\{A_v\}_{v \in V}$ be a finite family of sets and c a closure operator on V such that for any non-empty and non-closed subset X of V,*

$$\bigcap_{x \in X} A_x \;\subseteq\; \bigcup_{v \notin X} A_v\,. \tag{3.3}$$

Then, for any non-empty subset I of V,

$$\bigcap_{i \in I} A_i \;=\; \bigcap_{i \in c(I)} A_i\,.$$

Proof. Fix some $I \subseteq V$, $I \neq \emptyset$. There is nothing to prove if $\bigcap_{i\in I} A_i = \emptyset$. Otherwise choose $\omega \in \bigcap_{i\in I} A_i$ and show that $\omega \in \bigcap_{i\in c(I)} A_i$. By the choice of ω, $I \subseteq V_\omega$ where $V_\omega := \{v \in V \,|\, \omega \in A_v\}$. By the definition of V_ω and (3.3), V_ω is closed and hence, $c(I) \subseteq V_\omega$. Thus, $\omega \in \bigcap_{i\in c(I)} A_i$ and the proof is complete. □

We are now ready to state the main result of this section, which is both a generalization and improvement of the classical inclusion-exclusion identity.

Theorem 3.1.14 [Doh00b] *Let (V, c) be a convex geometry and $\{A_v\}_{v\in V}$ a finite family of sets such that for any non-empty and non-closed subset X of V,*

$$\bigcap_{x\in X} A_x \;\subseteq\; \bigcup_{v\notin X} A_v \,.$$

Then,

$$\chi\left(\bigcup_{v\in V} A_v\right) \;=\; \sum_{\substack{J\in\mathcal{P}^*(V)\\ J \text{ free}}} (-1)^{|J|-1}\,\chi\left(\bigcap_{j\in J} A_j\right) .$$

Proof. By the classical inclusion-exclusion identity (1.1) we have $\chi\left(\bigcup_{v\in V} A_v\right) = \sum_{I\subseteq V}(-1)^{|I|-1} g(I)$ where $g(I) := \chi\left(\bigcap_{i\in I} A_i\right)$ if $I \neq \emptyset$ and $g(\emptyset) := 0$. By Proposition 3.1.13 and since $c(\emptyset) = \emptyset$ we have $g = g \circ c$. Now apply Proposition 3.1.9. □

Remarks. Note that by setting $c(X) := X$ for any subset X of V, the improved identity of Theorem 3.1.14 specializes to the classical inclusion-exclusion identity.

The reader should also note that the requirements of Theorem 3.1.14 are satisfied if $\bigcap_{x\in X} A_x \subseteq A_v$ for any non-empty subset X of V and any $v \in c(X)$.

We further remark that the improved identity of Theorem 3.1.14 involves intersections of at most $h(c) := \max\{|J| : J \text{ } c\text{-free}\}$ sets. In [JW81] it is shown that $h(c)$ is the Helly number of the family of all c-closed subsets of V, that is, the smallest integer h such that any family of c-closed subsets of V whose intersection is empty has a subfamily of h or fewer sets whose intersection is also empty.

From Theorem 3.1.14 we now deduce some results, which for the first time appear in a common context. Among these results are the semilattice sieve of Narushima [Nar74, Nar77] and the tree sieve of Naiman and Wynn [NW92].

Corollary 3.1.15 [Nar74, Nar77] *Let $\{A_v\}_{v\in V}$ be a finite family of sets, where V is an upper semilattice such that $A_x \cap A_y \subseteq A_{x\vee y}$ for any $x, y \in V$. Then,*

$$\chi\left(\bigcup_{v\in V} A_v\right) \;=\; \sum_{\substack{I\in\mathcal{P}^*(V)\\ I \text{ is a chain}}} (-1)^{|I|-1}\,\chi\left(\bigcap_{i\in I} A_i\right) . \tag{3.4}$$

Proof. Apply Theorem 3.1.14 in connection with Example 3.1.7. □

Remark. Corollary 3.1.15 reduces to the classical inclusion-exclusion identity if V is a chain. The case where $A_x \cap A_y = A_{x \vee y}$ for any $x, y \in V$ is treated in [Doha].

The following result is due to Naiman and Wynn [NW92]:

Corollary 3.1.16 [NW92] *Let $\{A_v\}_{v \in V}$ be a finite family of sets, where the indices form the vertices of a tree $G = (V, E)$ such that $A_x \cap A_y \subseteq A_z$ for any $x, y \in V$ and any z on the unique path between x and y in G. Then,*

$$\chi\left(\bigcup_{v \in V} A_v\right) = \sum_{i \in V} \chi(A_i) - \sum_{\{i,j\} \in E} \chi(A_i \cap A_j)\,.$$

Proof. Apply Theorem 3.1.14 in connection with Example 3.1.5. □

Remark. The particular case where G is a path on n vertices gives rise to

$$\chi\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \chi(A_i) - \sum_{i=2}^{n} \chi(A_{i-1} \cap A_i)\,,$$

which is valid for all finite sequences of sets $A_1, \ldots, A_n$ that satisfy $A_i \cap A_j \subseteq A_k$ for $i, j = 1, \ldots, n$ and $k = i, \ldots, j$. This latter consequence is again due to Naiman and Wynn [NW92] and explains the cancellations in Figure 1.1.

Since any tree is a connected chordal graph, the following corollary generalizes the preceding one.

Corollary 3.1.17 [Doh99b] *Let $\{A_v\}_{v \in V}$ be a finite family of sets, and let $G = (V, E)$ be a connected chordal graph such that $A_x \cap A_y \subseteq A_z$ for any $x, y \in V$ and any z on any chordless path between x and y in G. Then,*

$$\chi\left(\bigcup_{v \in V} A_v\right) = \sum_{\substack{I \in \mathcal{P}^*(V) \\ I \text{ is a clique}}} (-1)^{|I|-1} \chi\left(\bigcap_{i \in I} A_i\right)\,. \tag{3.5}$$

Proof. We apply Theorem 3.1.14. For any subset X of V define

$$c(X) := \bigcup_{x,y \in X} \{z \in V \mid z \text{ is on a chordless path between } x \text{ and } y\}\,.$$

Then, (V, c) is a convex geometry, where a subset X of V is free if and only if X is a clique of G [EJ85, FJ86]. Theorem 3.1.14 now gives the result. □

Remarks. Note that Corollary 3.1.17 specializes to the classical inclusion-exclusion identity if G is complete, since then all subsets of the vertex-set are cliques.

Howorka [How81] showed that in chordal graphs having the property that all cycles of length five have at least three chords the chordless paths are precisely the shortest paths and that chordal graphs with this property are unions of Ptolemaic graphs and vice versa. Ptolemaic graphs are defined below. For a discussion of this class of graphs and other restricted classes of chordal graphs in connection with convex geometries, the reader is referred to Farber and Jamison [FJ86].

Definition 3.1.18 A graph G is called *Ptolemaic* if it is connected and if for any four vertices v, w, x, y of G, $d(v,w)d(x,y) \leq d(v,x)d(w,y) + d(w,x)d(v,y)$ where $d(a,b)$ denotes the length of a shortest path between a and b in G.

By Corollary 3.1.17 and the preceding remarks, we have the following result:

Corollary 3.1.19 [Doh99b] *Let $\{A_v\}_{v\in V}$ be a finite family of sets, and let $G = (V,E)$ be a Ptolemaic graph such that $A_x \cap A_y \subseteq A_z$ for any $x, y \in V$ and any z on any shortest path between x and y in G. Then, (3.5) holds.*

We proceed with deducing some further consequences of Theorem 3.1.14.

Corollary 3.1.20 *Let $\{A_v\}_{v\in V}$ be a finite family of sets where for any non-empty subset X of V there is a unique minimal non-empty subset Y of X such that*

$$\bigcap_{x\in X} A_x \;=\; \bigcap_{y\in Y} A_y\,. \tag{3.6}$$

Then,

$$\chi\left(\bigcup_{v\in V} A_v\right) \;=\; \sum_{I} (-1)^{|I|-1}\,\chi\left(\bigcap_{i\in I} A_i\right)\,,$$

where the sum is extended over all non-empty subsets I of V such that $\bigcap_{i\in I} A_i \neq \bigcap_{j\in J} A_j$ for all non-empty proper subsets and proper supersets J of I.

Proof. Again, we apply Theorem 3.1.14. It is straightforward to check that

$$c(X) \;:=\; \left\{ v\in V \;\middle|\; \bigcap_{x\in X} A_x \subseteq A_v \right\} \quad (X\neq\emptyset)\,; \qquad c(\emptyset) \;:=\; \emptyset\,;$$

defines a closure operator on V. In order to verify the unique basis property of Definition 3.1.3, let X be a non-empty closed subset of V, and let Y be the unique minimal non-empty subset of X satisfying (3.6). Then, $c(X) = c(Y) \subseteq X$ and hence, $c(Y) = X$. Now, to show that Y is smallest with respect to $c(Y) = X$, suppose that $c(Y') = X$ for some non-empty subset Y' of X. Then, $\bigcap_{x\in X} A_x = \bigcap_{y\in Y'} A_y$ and hence, $Y' \supseteq Y$ by the choice of Y. Thus, Y is the unique c-basis of X. The description of the free sets directly follows from Proposition 3.1.2. □

The dual version of the following corollary has been published in [Doh97] together with a generalization to Möbius inversion over power set lattices.

Corollary 3.1.21 [Doh97] *Let $\{A_v\}_{v\in V}$ be a finite family of sets, where V is endowed with a linear ordering relation, and let $\mathfrak{X}$ be a set of non-empty subsets of V such that for any $X \in \mathfrak{X}$,*

$$\bigcap_{x\in X} A_x \;\subseteq\; A_v \quad \textit{for some } v > \max X.$$

Then,

$$\chi\left(\bigcup_{v\in V} A_v\right) \;=\; \sum_{\substack{I\in\mathcal{P}^*(V)\\ I\not\supseteq X\,(\forall X\in\mathfrak{X})}} (-1)^{|I|-1}\,\chi\left(\bigcap_{i\in I} A_i\right).$$

Proof. By the requirements of the corollary there is a family $\{v_X\}_{X\in\mathfrak{X}} \subseteq V$ such that for any $X \in \mathfrak{X}$, $\bigcap_{x\in X} A_x \subseteq A_{v_X}$ where $v_X > \max X$. For any $I \subseteq V$ define

$$c(I) \;:=\; I \cup \{v_X \,|\, X \in \mathfrak{X},\, X \subseteq I\}$$

as well as

$$c^*(I) \;:=\; c(I) \cup c(c(I)) \cup c(c(c(I))) \cup c(c(c(c(I)))) \cup \cdots.$$

Then, c^* is a closure operator on V. Since $v_X > \max X$ for any $X \in \mathfrak{X}$ we find that $I \setminus \{v_X \,|\, X \in \mathfrak{X},\, X \subseteq I\}$ is the unique c^*-basis of any c^*-closed subset I of V. Thus, (V, c^*) is a convex geometry. Evidently, a subset I of V is c^*-free if and only if $I \not\supseteq X$ for any $X \in \mathfrak{X}$. Now apply Theorem 3.1.14 with c^* instead of c. □

Remark. Note that the dual version of the preceding corollary is obtained by replacing $v > \max X$ with $v < \min X$. The reader should also note that the corollary reduces to the classical inclusion-exclusion principle if $\mathfrak{X}$ is empty.

As a consequence of Corollary 3.1.21 we deduce the following generalization of Corollary 3.1.15, which is again due to Narushima [Nar82].

Corollary 3.1.22 [Nar82] *Let $\{A_v\}_{v\in V}$ be a finite family of sets, where V is endowed with a partial ordering relation such that for any $x, y \in V$, $A_x \cap A_y \subseteq A_z$ for some upper bound z of x and y. Then, (3.4) holds.*

Proof. Corollary 3.1.22 follows from Corollary 3.1.21 by defining $\mathfrak{X}$ as the set of all unordered pairs of incomparable elements of V and then considering an arbitrary linear extension of the partial ordering relation on V. □

Remark. Note that the requirements of Corollary 3.1.22 are weaker than the original requirements of Narushima [Nar82]. Namely, Narushima requires that for any $x, y \in V$, $A_x \cap A_y \subseteq A_z$ for some *minimal* upper bound z of x and y. In

Corollary 3.1.22, however, the minimality of z is not required. From this point of view, Corollary 3.1.22 is more general than Narushima's original result [Nar82].

We close this section with a generalization of Proposition 3.1.8, Proposition 3.1.9 and Theorem 3.1.14 to the situation where the closure operator does not have the unique basis property. To this end, we need the following definition:

Definition 3.1.23 Let V be a finite set and c a closure operator on V. A *c-formation* of a subset X of V is any non-empty set $\mathcal{B}$ of c-bases of X such that $\bigcup \mathcal{B} = X$. A c-formation $\mathcal{B}$ of X is *odd* resp. *even* if $|\mathcal{B}|$ is odd resp. even. The *c-domination* of X, $\mathrm{dom}_c(X)$, is the number of odd c-formations of X minus the number of even c-formations of X. Evidently, if X is not c-closed, then $\mathrm{dom}_c(X) = 0$, and if X is c-free, then $\mathrm{dom}_c(X) = 1$. If (V, c) is a convex geometry, then $\mathrm{dom}_c(X) = 1$ resp. 0 depending on whether X is c-free or not.

Proposition 3.1.24 [Doh99b] *Let V be a finite set and c a closure operator on V. Then, for any c-closed subset J of V,*

$$\sum_{\substack{I \subseteq J \\ c(I) = J}} (-1)^{|I|} \;=\; (-1)^{|J|} \,\mathrm{dom}_c(J)\,.$$

Proof. Let $J_0, \ldots, J_n$ be the distinct bases of J. Evidently, $c(I) = J$ if and only if $J_k \subseteq I \subseteq J$ for some $k \in \{0, \ldots, n\}$. Thus, by including and excluding terms,

$$\sum_{\substack{I \subseteq J \\ c(I) = J}} (-1)^{|I|} = \sum_{\substack{\mathcal{B} \subseteq \{J_0, \ldots, J_n\} \\ \mathcal{B} \neq \emptyset}} (-1)^{|\mathcal{B}|-1} \sum_{I \,:\, \bigcup \mathcal{B} \subseteq I \subseteq J} (-1)^{|I|} = \sum_{\substack{\mathcal{B} \subseteq \{J_0, \ldots, J_n\} \\ \mathcal{B} \neq \emptyset,\, \bigcup \mathcal{B} = J}} (-1)^{|\mathcal{B}|-1} (-1)^{|J|}\,.$$

The result now follows from the definition of the c-domination $\mathrm{dom}_c(J)$. □

Proposition 3.1.25 [Doh99b] *Let V be a finite set, c a closure operator on V and g a mapping from $\mathcal{P}(V)$ into an abelian group such that $g = g \circ c$. Then,*

$$\sum_{I \subseteq V} (-1)^{|I|}\, g(I) \;=\; \sum_{\substack{J \subseteq V \\ J \text{ closed}}} (-1)^{|J|}\, \mathrm{dom}_c(J)\, g(J)\,.$$

Proof. Proposition 3.1.25 follows from Proposition 3.1.24 in the same way as Proposition 3.1.9 follows from Proposition 3.1.8. □

Theorem 3.1.26 [Doh99b] *Let $\{A_v\}_{v \in V}$ be a finite family of sets and c a closure operator on V such that for any non-empty and non-closed subset X of V,*

$$\bigcap_{x \in X} A_x \;\subseteq\; \bigcup_{v \notin X} A_v$$

and such that $c(\emptyset) = \emptyset$. *Then,*

$$\chi\left(\bigcup_{v\in V} A_v\right) = \sum_{\substack{J\in\mathcal{P}^*(V)\\ J \text{ closed}}} (-1)^{|J|-1}\,\mathrm{dom}_c(J)\,\chi\left(\bigcap_{j\in J} A_j\right).$$

Proof. Theorem 3.1.26 follows from Proposition 3.1.13 and Proposition 3.1.25 like Theorem 3.1.14 follows from Proposition 3.1.13 and Proposition 3.1.9. □

3.2 Improvements based on kernel operators

Similar inclusion-exclusion results as for closure operators having the unique basis property are now established for kernel operators. There is, however, no duality between the results of the preceding section and the results of the present section.

Definition 3.2.1 Let V be a set. A *kernel operator* on V is a mapping k from the power set of V into itself such that for all subsets X and Y of V,

$$\begin{aligned} &\text{(i)} \quad k(X) \subseteq X \quad \text{(intensionality)},\\ &\text{(ii)} \quad X \subseteq Y \;\Rightarrow\; k(X) \subseteq k(Y) \quad \text{(monotonicity)},\\ &\text{(iii)} \quad k(k(X)) = k(X) \quad \text{(idempotence)}. \end{aligned}$$

If k is a kernel operator on V, then a subset X of V is called *k-open* if $k(X) = X$.

There is a well-known correspondence between kernel operators on V and union-closed subsets of the power set of V. Similarly, there is a correspondence between closure operators and intersection-closed subsets, which shall not be of interest to us. For a proof of the following proposition, we refer to Erné [Ern82].

Proposition 3.2.2 [Ern82] *Let V be a finite set. If k is a kernel operator on V, then $\{X \subseteq V \mid X$ k-open$\}$ is union-closed. If $\mathcal{X} \subseteq \mathcal{P}(V)$ is union-closed, then*

$$I \;\mapsto\; \bigcup\{X \in \mathcal{X} \mid X \subseteq I\} \qquad (I \subseteq V)$$

defines a kernel operator k on V such that X is k-open if and only if $X \in \mathcal{X}$.

Lemma 3.2.3 [Doh99f] *Let V be a set, k a kernel operator on V and X_0 a k-open subset of V. Then, all minimal sets in $\{Y \subseteq V \mid k(Y) \supset X_0\}$ are k-open.*

Proof. Assume that Y is not k-open and $k(Y) \supset X_0$. Then, $k(Y) \subset Y$ and $k(k(Y)) \supset X_0$, thus showing that Y is not minimal in $\{Y \subseteq V \mid k(Y) \supset X_0\}$. □

The following theorem will be used below to prove the main result of this section. It has applications not only to inclusion-exclusion, but also to the Tutte polynomial, the β invariant and the Möbius function (see Sections 6.3 and 6.4).

Theorem 3.2.4 [Doh99f] *Let V be a finite set, and let f and g be mappings from the power set of V into an abelian group such that $f(X) = \sum_{Y \supseteq X} g(Y)$ for any subset X of V. Furthermore, let k be a kernel operator on V, and let X_0 be a k-open subset of V such that $f(X) = 0$ for any k-open $X \supset X_0$. Then,*

$$f(X_0) \;=\; \sum_{Y\,:\,k(Y)=X_0} g(Y)\,.$$

Proof. Obviously, we only have to show that $\sum_{Y\,:\,k(Y)\supset X_0} g(Y) = 0$. Let $\mathcal{Y}_0$ consist of all subsets Y of V which satisfy $k(Y) \supset X_0$ and which are minimal with respect to this property. By including and excluding terms we find that

$$\begin{aligned}\sum_{Y\,:\,k(Y)\supset X_0} g(Y) \;&=\; \sum_{\substack{Y\,\supseteq\, Z \text{ for}\\ \text{some } Z\,\in\,\mathcal{Y}_0}} g(Y) \;=\; \sum_{\mathcal{Z}\subseteq\mathcal{Y}_0,\,\mathcal{Z}\neq\emptyset} (-1)^{|\mathcal{Z}|-1} \sum_{\substack{Y\,\supseteq\, Z \text{ for}\\ \text{any } Z\,\in\,\mathcal{Z}}} g(Y)\\ &=\; \sum_{\mathcal{Z}\subseteq\mathcal{Y}_0,\,\mathcal{Z}\neq\emptyset} (-1)^{|\mathcal{Z}|-1} \sum_{Y\supseteq\bigcup\mathcal{Z}} g(Y) \;=\; \sum_{\mathcal{Z}\subseteq\mathcal{Y}_0,\,\mathcal{Z}\neq\emptyset} (-1)^{|\mathcal{Z}|-1} f\big(\bigcup\mathcal{Z}\big)\,.\end{aligned}$$

By the preceding lemma, any $Z \in \mathcal{Z}$ is k-open and hence, $\bigcup\mathcal{Z}$ is k-open. Since obviously $\bigcup\mathcal{Z} \supset X_0$, the requirements give $f\big(\bigcup\mathcal{Z}\big) = 0$, whence the result. □

The main result of this section is the following:

Theorem 3.2.5 [Doh99f] *Let $\{A_v\}_{v\in V}$ be a finite family of sets, and let k be a kernel operator on V such that for any non-empty and k-open subset X of V,*

$$\bigcap_{x\in X} A_x \;\subseteq\; \bigcup_{v\notin X} A_v\,.$$

Then,

$$\chi\left(\bigcup_{v\in V} A_v\right) \;=\; \sum_{\substack{I\,\in\,\mathcal{P}^*(V)\\ k(I)=\emptyset}} (-1)^{|I|-1}\,\chi\left(\bigcap_{i\in I} A_i\right).$$

Subsequently, we give two different proofs of Theorem 3.2.5. The first proof combines Theorem 3.2.4 with the classical inclusion-exclusion principle, whereas the second proof generalizes Garsia and Milne's bijective proof of the classical inclusion-exclusion principle [GM81, Zei84, Pau86] as well as Blass and Sagan's bijective proof of Whitney's broken circuit theorem on chromatic polynomials [BS86, Whi32]. We come back to Whitney's theorem in Section 6.2, p. 91.

First proof. We apply Theorem 3.2.4. For any subsets X and Y of V define

$$f(X) \;:=\; (-1)^{|X|}\,\chi\left(\bigcap_{x\in X} A_x \cap \bigcap_{v\notin X} \complement A_v\right);\quad g(Y) \;:=\; (-1)^{|Y|}\,\chi\left(\bigcap_{y\in Y} A_y\right)$$

where, by convention, $\bigcap_{x\in\emptyset} A_x = \bigcap_{x\notin V} \complement A_v = \Omega$ (the ground set). By applying the principle of inclusion-exclusion we find that $f(X) = \sum_{Y\supseteq X} g(Y)$ for any subset X of V, and by the requirements of the theorem, $f(X) = 0$ for any non-empty and k-open subset X of V. Hence, by applying Theorem 3.2.4, we obtain $\chi\big(\bigcap_{v\in V} \complement A_v\big) = \sum_{I:k(I)=\emptyset}(-1)^{|I|}\chi\big(\bigcap_{i\in I} A_i\big)$, from which the statement of the theorem follows since, due to a law of De Morgan, $\chi\left(\bigcap \complement A_v\right) = 1-\chi\big(\bigcup A_v\big)$. □

Second proof. Evidently, it suffices to prove that

$$\chi\left(\bigcup_{v\in V} A_v\right) + \sum_{\substack{I\in\mathcal{P}^*(V)\\ k(I)=\emptyset\\ |I| \text{ even}}} \chi\left(\bigcap_{i\in I} A_i\right) = \sum_{\substack{I\in\mathcal{P}^*(V)\\ k(I)=\emptyset\\ |I| \text{ odd}}} \chi\left(\bigcap_{i\in I} A_i\right). \tag{3.7}$$

For any $\omega \in \bigcup_{v\in V} A_v$ define $V_\omega := \{v \in V \,|\, \omega \in A_v\}$ as well as

$$\begin{aligned}\mathcal{E}(\omega) &:= \big\{I \in \mathcal{P}(V_\omega) \,\big|\, k(I) = \emptyset,\ |I| \text{ even}\big\},\\ \mathcal{O}(\omega) &:= \big\{I \in \mathcal{P}(V_\omega) \,\big|\, k(I) = \emptyset,\ |I| \text{ odd}\big\}.\end{aligned}$$

Then, (3.7) is equivalent to

$$|\mathcal{E}(\omega)| = |\mathcal{O}(\omega)| \quad \text{for all } \omega \in \bigcup_{v\in V} A_v\,. \tag{3.8}$$

Now, in order to prove (3.8), fix some $\omega \in \bigcup_{v\in V} A_v$. By the definition of V_ω and the requirements of the theorem we find that V_ω is not k-open, whence some $v \in V_\omega \setminus k(V_\omega)$ can be chosen. It follows that for any subset I of V_ω, $v \notin k(I\cup\{v\})$ since otherwise $v \in k(I \cup \{v\}) \subseteq k(V_\omega \cup \{v\}) = k(V_\omega)$, contradicting $v \notin k(V_\omega)$. Since $v \notin k(I \cup \{v\})$ and $k(I \cup \{v\}) \subseteq I \cup \{v\}$ we obtain $k(I \cup \{v\}) \subseteq I$ and hence, $k(I\cup\{v\}) \subseteq k(I)$. From the latter we conclude that for any subset I of V_ω, $k(I) = \emptyset \Rightarrow k(I \cup \{v\}) = \emptyset$. Hence, $I \mapsto I \vartriangle \{v\}$, where $\vartriangle$ denotes symmetric difference, is a bijective mapping from $\mathcal{E}(\omega)$ to $\mathcal{O}(\omega)$. Thus, (3.8) is shown. □

Remarks. Note that by setting $k(I) := \emptyset$ for any subset I of V, Theorem 3.2.5 specializes to the classical inclusion-exclusion identity. Similarly, the following theorems specialize to the classical identity if $\mathfrak{X}$ is chosen to be empty.

Note that by the correspondence between kernel operators and union-closed sets, the following theorem is equivalent to the preceding one.

Theorem 3.2.6 [Doh00e] *Let $\{A_v\}_{v\in V}$ be a finite family of sets, and let $\mathfrak{X}$ be a union-closed set of non-empty subsets of V such that for any $X \in \mathfrak{X}$,*

$$\bigcap_{x\in X} A_x \subseteq \bigcup_{v\notin X} A_v\,.$$

Then,

$$\chi\left(\bigcup_{v\in V} A_v\right) \;=\; \sum_{\substack{I\in\mathcal{P}^*(V)\\ I\not\supseteq X(\forall X\in\mathcal{X})}} (-1)^{|I|-1}\,\chi\left(\bigcap_{i\in I} A_i\right). \tag{3.9}$$

Proof. The result follows by combining Proposition 3.2.2 and Theorem 3.2.5. □

Corollary 3.2.7 [Dohc] *Let $\{A_v\}_{v\in V}$ be a finite family of sets, c a closure operator on V and $\mathcal{X}$ a set of non-empty subsets of V such that $\{c(X)\,|\,X\in\mathcal{X}\}$ is a chain and such that for any $X\in\mathcal{X}$,*

$$\bigcap_{x\in X} A_x \;\subseteq\; \bigcup_{v\notin c(X)} A_v\,. \tag{3.10}$$

Then, (3.9) holds.

Proof. Apply Theorem 3.2.6 with $\mathcal{X}' := \{\bigcup\mathcal{I}\,|\,\emptyset\neq\mathcal{I}\subseteq\mathcal{X}\}$ instead of $\mathcal{X}$. Then, $\mathcal{X}'$ is union-closed, and for any $X'\in\mathcal{X}'$ the requirements of the corollary imply

$$X' \;=\; \bigcup\{X\,|\,X\in\mathcal{X},\,X\subseteq X'\} \;\subseteq\; \bigcup\{c(X)\,|\,X\in\mathcal{X},\,X\subseteq X'\} \;=\; c(X_0)$$

for some $X_0\in\mathcal{X}$, $X_0\subseteq X'$. It follows that $X_0\subseteq X'\subseteq c(X_0)$ and therefore,

$$\bigcap_{x\in X'} A_x \;\subseteq\; \bigcap_{x\in X_0} A_x \;\subseteq\; \bigcup_{v\notin c(X_0)} A_v \;\subseteq\; \bigcup_{v\notin X'} A_v\,,$$

whence Theorem 3.2.6 gives the result. □

The following result generalizes Corollary 3.1.21. For an exemplary application of this result to counting arrangements with forbidden positions on a chess-like board, see [Doh99a]. See also [Doh00c] for some additional remarks.

Corollary 3.2.8 [Doh99a] *Let $\{A_v\}_{v\in V}$ be a finite family of sets, where V is endowed with a linear ordering relation, and let $\mathcal{X}$ be a set of non-empty subsets of V such that for any $X\in\mathcal{X}$,*

$$\bigcap_{x\in X} A_x \;\subseteq\; \bigcup_{v>\max X} A_v\,. \tag{3.11}$$

Then, (3.9) holds.

Proof. Corollary 3.2.8 follows from Corollary 3.2.7 by using the closure operator $X \mapsto c(X)$ where $c(X) := \{v \in V \mid v \leq \max X\}$ if $X \neq \emptyset$, and $c(\emptyset) := \emptyset$. $\square$

We close this section with two self-contained proofs of Corollary 3.2.7 and Corollary 3.2.8, which only require the traditional inclusion-exclusion principle.

Alternative proof of Corollary 3.2.7. [Dohc] By the traditional form of the inclusion-exclusion principle, it suffices to prove that

$$\sum_{I \in \overline{\mathfrak{X}}} (-1)^{|I|-1} \chi\left(\bigcap_{i \in I} A_i\right) \;=\; 0\,, \tag{3.12}$$

where

$$\overline{\mathfrak{X}} \;:=\; \{I \subseteq V \mid I \neq \emptyset \text{ and } I \supseteq X \text{ for some } X \in \mathfrak{X}\}\,. \tag{3.13}$$

The proof employs the following partial ordering relation on the power set of V:

$$A \subseteq_c B \;:\Leftrightarrow\; c(A) \cap B = A \quad \text{for any } A, B \subseteq V.$$

Of course, $\subseteq_c$ is reflexive $[A \subseteq c(A) \Rightarrow c(A) \cap A = A \Rightarrow A \subseteq_c A]$, antisymmetric $[(A \subseteq_c B) \wedge (B \subseteq_c A) \Rightarrow (c(A) \cap B = A) \wedge (c(B) \cap A = B) \Rightarrow A = c(A) \cap B = c(A) \cap c(B) \cap A = c(B) \cap A = B]$ and transitive $[(A \subseteq_c B) \wedge (B \subseteq_c C) \Rightarrow (c(A) \cap B = A) \wedge (c(B) \cap C = B) \Rightarrow c(A) \cap C = c(c(A) \cap B) \cap C \subseteq c(A) \cap c(B) \cap C = c(A) \cap B = A \Rightarrow A \subseteq_c C]$. In order to utilize this partial ordering relation in proving (3.12), we first show that for any $Y_1, Y_2 \in \overline{\mathfrak{X}}$ which are $\subseteq_c$-minimal in $\overline{\mathfrak{X}}$,

$$\{I \subseteq V \mid I \supseteq_c Y_1\} \cap \{I \subseteq V \mid I \supseteq_c Y_2\} \neq \emptyset \;\Rightarrow\; Y_1 = Y_2\,. \tag{3.14}$$

To this end, let $I \supseteq_c Y_1$ and $I \supseteq_c Y_2$. Then, by definition of $\subseteq_c$, $Y_1 = I \cap c(Y_1)$ $(\subseteq I)$ and $Y_2 = I \cap c(Y_2)$ $(\subseteq I)$. It follows that $c(c(Y_1) \cap Y_2) \cap Y_1 \subseteq c(Y_1) \cap c(Y_2) \cap Y_1 = Y_1 \cap c(Y_2) = I \cap c(Y_1) \cap c(Y_2) = c(Y_1) \cap Y_2 = c($